LOCAL HEATING OF A SEMIINFINITE BODY BY A LASER SOURCE

V. P. Kozlov

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The authors have determined the laws for development of spatial unsteady temperature fields in a semiinfinite nontransparent body heated by a Gaussian laser beam.

In a thermophysical experiment one finds electron [1] and laser [2] sources of local heating of a body surface as original and efficient methods of generating a directed heat flux. These new sources for generating a heat flux have now a secure place in the practice of investigating thermophysical properties of substances. In particular, reference [1] has described various methods of applying electron heating in calorimetric measurements of heat capacity, integral emissivity, heat conduction and other heat transfer characteristics.

The aim of this paper is to describe analytically the laws for development of spatial unsteady temperature fields in a semiinfinite body whose surface is subject to the local action of laser radiation. We shall assume that the power of the laser radiation $W_0(\tau)$ incident on the body surface not only does not disintegrate the material, but also does not cause nonlinear variation of the thermophysical properties with temperature in the vicinity of the heated spot.

Most laser beams of diameter $2r_0$ have azimuthal symmetry in the transverse section, with the greatest intensity on the axis. With increase of distance r from the beam axis the intensity $w_0(\tau) = W_0(\tau)/(\pi r_0^2)$ falls off according to an exponential law, i.e.,

$$w_0(r,\ \tau) = w_0(\tau) \exp\left(-\frac{r^2}{r_0^2}\right), \tag{1}$$

where $w_0(r, \tau)$ is the intensity (density) of the laser radiation (W/m^2).

The value of r for which the radiative intensity (the energy per unit area $S = \pi r_0^2$) decreases by a factor of e compared with the intensity on the beam axis is called the transverse dimension of the beam, r_0 [2]. Generally speaking, a r_0 varies from point to point along the beam axis. At some point within the resonator, called the neck of the beam, a Gaussian beam has a minimum dimension $r_{0\,\min}$. Below we shall assume that a beam of diameter $2r_0$, is focused (normally) on the body surface, and that its intensity on the body surface varies according to the law (1). However, the density of heat flux absorbed by the body itself (more accurately by its planar boundary surface) will depend on the emissivity of the body surface, characterized by the emittance or the absorptance. Therefore, we write a boundary condition of the second kind at the surface ($z = 0,\ 0 \le r \le r_0$) in the form

$$-\lambda \left.\frac{\partial \Theta_1(r,\ 0,\ \tau)}{\partial z}\right|_{\substack{0\le r\le r_0,\\ z=0}} = \rho w_0(\tau) \exp\left(-\frac{r^2}{r_0^2}\right), \tag{2}$$

where $\rho \le 1$ is a dimensionless parameter characterizing the absorptance of the body (the equal sign can be assumed for a perfect blackbody [3]).

In the region of representations of the excess temperature $\bar{\Theta}_{ic}(r, p, s)$ of a semiinfinite body (where p and s are, respectively, the parameters of the infinite Fourier cosine and Laplace transformations, $i = \overline{1, 2}$) one must find the solution of the following system of differential equations (we choose the origin of the cylindrical coordinates at the central point $r = z = 0$ of the circular heated spot on the body surface) [4]:

$$\frac{\partial^2 \bar{\Theta}_{1c}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Theta}_{1c}}{\partial r} - \left(p^2 + \frac{s}{a}\right)\bar{\Theta}_{1c} = -\frac{\rho \bar{w}_0(s)}{\lambda} \exp\left(-\frac{r^2}{r_0^2}\right) (0 \le r < r_0), \tag{3}$$

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$$\frac{\partial^2\overline{\Theta}_{2c}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{\Theta}_{2c}}{\partial r}-\left(p^2+\frac{s}{a}\right)\overline{\Theta}_{2c}=0 \ (r>r_0) \quad (4)$$

with the boundary conditions

$$\overline{\Theta}_{1c}(r_0,\ p,\ s)=\overline{\Theta}_{2c}(r_0,\ p,\ s), \quad (5)$$

$$\frac{\partial\overline{\Theta}_{1c}(r_0,\ p,\ s)}{\partial r}=\frac{\partial\overline{\Theta}_{2c}(r_0,\ p,\ s)}{\partial r}, \quad (6)$$

$$\frac{\partial\overline{\Theta}_{1c}(0,\ p,\ s)}{\partial r}=0, \quad (7)$$

$$\frac{\partial\overline{\Theta}_{2c}(\infty,\ p,\ s)}{\partial r}=0. \quad (8)$$

The general integrals of Eqs. (3) and (4) can be written in the form [4]

$$\overline{\Theta}_{1c}(r,\ p,\ s)=C_1(r)I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)+C_2(r)K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)(r<r_0), \quad (9)$$

$$\overline{\Theta}_{2c}(r,\ p,\ s)=C_3I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)+C_4K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)(r>r_0), \quad (10)$$

where $I_0(X)$, $K_0(X)$ are modified Bessel functions.

We find the constants of integration $C_1(r)$ and $C_2(r)$ by the method of variation of constants [5]:

$$C_1(r)=-\frac{\rho\overline{w}_0(s)}{\lambda}\int_0^r x\exp\left(-\frac{x^2}{r_0^2}\right)K_0\left(x\sqrt{p^2+\frac{s}{a}}\right)dx+B_1, \quad (11)$$

$$C_2(r)=\frac{\rho\overline{w}_0(s)}{\lambda}\int_0^r x\exp\left(-\frac{x^2}{r_0^2}\right)I_0\left(x\sqrt{p^2+\frac{s}{a}}\right)dx+B_2,$$

where B_1 and B_2 are constants of integration; and $\overline{w}_0(s)$ is the Laplace-transformed value of the specific power (per unit area) of the laser source.

Using the boundary conditions at $r=0$ and $r=\infty$ we find that the constants $C_3=B_2=0$. Thus, the solutions for $\overline{\Theta}_{1c}(r,\ p,\ s)$ and $\overline{\Theta}_{2c}(r,\ p,\ s)$ take the following form:

$$\overline{\Theta}_{1c}=B_1I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)-\frac{\rho\overline{w}_0(s)}{\lambda}\left\{I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)\times\right.$$
$$\times\int_0^r x\exp\left(-\frac{x^2}{r_0^2}\right)K_0\left(x\sqrt{p^2+\frac{s}{a}}\right)dx-K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)\times$$
$$\left.\times\int_0^r x\exp\left(-\frac{x^2}{r_0^2}\right)I_0\left(x\sqrt{p^2+\frac{s}{a}}\right)dx\right\}(r<r_0); \quad (12)$$

$$\overline{\Theta}_{2c}=C_4K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)(r>r_0). \quad (13)$$

Using the matching conditions (equality) of temperatures and their gradients on the cylindrical surface $r=r_0$ inside the body, we find that

$$B_1=-\frac{r_0\rho\overline{w}_0(s)K_1\left(r_0\sqrt{p^2+\frac{s}{a}}\right)}{e\lambda\sqrt{p^2+\frac{s}{a}}}-\frac{2\rho\overline{w}_0(s)}{\lambda r_0^2\sqrt{p^2+\frac{s}{a}}}\times\int_0^{r_0}x^2K_1\left(x\sqrt{p^2+\frac{s}{a}}\right)\exp\left(-\frac{x^2}{r_0^2}\right)dx, \quad (14)$$

$$C_4=\frac{r_0\rho\bar{w}_0(s)\,I_1\left(r_0\sqrt{p^2+\frac{s}{a}}\right)}{e\lambda\sqrt{p^2+\frac{s}{a}}}+\frac{2\rho\bar{w}_0(s)}{\lambda r_0^2\sqrt{p^2+\frac{s}{a}}}\times\int_0^{r_0}x^2I_1\left(x\sqrt{p^2+\frac{s}{a}}\right)\exp\left(-\frac{x^2}{r_0^2}\right)dx.$$

Substituting the values of the contants B_1 and C_4 into the desired solutions and applying the conversion formula for a cosine Fourier transformation, we obtain the following expressions for the representations (only the Laplace) of the excess temperatures $\bar{\Theta}_1(r, z, s)$ and $\bar{\Theta}_2(r, z, s)$:

$$\bar{\Theta}_1(r,\ z,\ s)=\frac{\rho\bar{w}_0(s)}{b\sqrt{s}}\exp\left(-\frac{r^2}{r_0^2}\right)\exp\left(-\frac{z}{\sqrt{a}}\sqrt{s}\right)-$$

$$-\frac{2r_0\rho\bar{w}_0(s)}{\pi e\lambda}\int_0^\infty\frac{\cos pz}{\sqrt{p^2+\frac{s}{a}}}K_1\left(r_0\sqrt{p^2+\frac{s}{a}}\right)I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)dp-$$

$$-\frac{4\rho\bar{w}_0(s)}{\pi\lambda r_0^2}\int_0^\infty\frac{\cos pz}{\sqrt{p^2+\frac{s}{a}}}I_0\left(r\sqrt{p^2+\frac{s}{a}}\right)\times\int_r^{r_0}x^2K_1\left(x\sqrt{p^2+\frac{s}{a}}\right)\exp\left(-\frac{x^2}{r_0^2}\right)dxdp+$$

$$+\frac{4\rho\bar{w}_0(s)}{\pi\lambda r_0^2}\int_0^\infty\frac{\cos pz}{\sqrt{p^2+\frac{s}{a}}}K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)\times\int_0^r x^2I_1\left(x\sqrt{p^2+\frac{s}{a}}\right)\exp\left(-\frac{x^2}{r_0^2}\right)dxdp\ (0\leqslant r\leqslant r_0);\tag{15}$$

$$\bar{\Theta}_2(r,\ z,\ s)=\frac{2r_0\rho\bar{w}_0(s)}{\pi e\lambda}\int_0^\infty\frac{\cos pz}{\sqrt{p^2+\frac{s}{a}}}K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)\times I_1\left(r_0\sqrt{p^2+\frac{s}{a}}\right)dp+\frac{4\rho\bar{w}_0(s)}{\pi\lambda r_0^2}\int_0^\infty\frac{\cos pz}{\sqrt{p^2+\frac{s}{a}}}\times$$

$$\times K_0\left(r\sqrt{p^2+\frac{s}{a}}\right)\int_0^{r_0}x^2I_1\left(x\sqrt{p^2+\frac{s}{a}}\right)\exp\left(-\frac{x^2}{r_0^2}\right)dxdp\ \ (r\geqslant r_0).\tag{16}$$

For our further investigations we shall be interested in the solution for $\Theta_1(0, z, \tau)$ on the axis $r = 0$ $(z \geq 0,\ \tau > 0)$, and also the solutions for $\Theta_1(r, 0, \tau)$ and $\Theta_2(r, 0, \tau)$ at $z = 0$ $(r \geq 0,\ \tau > 0)$, which can be written in the following form:

$$\Theta_1(0,\ z,\ \tau)=L^{-1}[\bar{\Theta}_1(0,\ z,\ s)]=\frac{\rho}{b}L^{-1}\left[\bar{w}_0(s)\frac{1}{\sqrt{s}}\times\right.$$

$$\left.\times\exp\left(-\frac{z}{\sqrt{a}}\sqrt{s}\right)\right]-\frac{\rho}{eb}L^{-1}\left[\bar{w}_0(s)\frac{1}{\sqrt{s}}\times\right.$$

$$\left.\times\exp\left(-\frac{\sqrt{r_0^2+z^2}}{\sqrt{a}}\sqrt{s}\right)\right]+\frac{2\rho}{b}\exp\left(\frac{z^2}{r_0^2}\right)L^{-1}\left\{\frac{\bar{w}_0(s)}{\sqrt{s}}\times\right.$$

$$\times\exp\left(\frac{sr_0^2}{4a}\right)\left[\frac{r_0\sqrt{s}}{2\sqrt{a}}\int_{\frac{z}{r_0}+\frac{r_0\sqrt{s}}{2\sqrt{a}}}^{\frac{\sqrt{r_0^2+z^2}}{r_0}+\frac{r_0\sqrt{s}}{2\sqrt{a}}}\exp(-t^2)\,dt-\right.$$

$$\left.\left.-\int_{\frac{z}{r_0}+\frac{r_0\sqrt{s}}{2\sqrt{a}}}^{\frac{\sqrt{r_0^2+z^2}}{r_0}+\frac{r_0\sqrt{s}}{2\sqrt{a}}}t\exp(-t^2)\,dt\right]\right\}=\frac{\sqrt{\pi}}{2}\frac{\rho r_0}{\lambda}\exp\left(\frac{z^2}{r_0^2}\right)\times\tag{17}$$

$$\times L^{-1}\left\{\bar{w}_0(s)\exp\left(\frac{sr_0^2}{4a}\right)\left[\operatorname{erf}\left(\frac{\sqrt{r_0^2+z^2}}{r_0}+\frac{r_0\sqrt{s}}{2\sqrt{a}}\right)-\right.\right.$$

$$- \operatorname{erf}\left(\frac{z}{r_0} + \frac{r_0 \sqrt{s}}{2\sqrt{a}}\right)\Big]\Big\} = \frac{\rho r_0^2}{4\sqrt{\pi}\lambda\sqrt{a}} \int_0^\tau \frac{w_0(\tau-\xi)}{\sqrt{\xi}\left(\xi + \frac{r_0^2}{4a}\right)} \times$$

$$\times \exp\left(-\frac{z^2}{4a\xi}\right)\left[1 - \frac{1}{e}\exp\left(-\frac{r_0^2}{4a\xi}\right)\right] d\xi,$$

since

$$L^{-1}\left\{\exp(\alpha s)\operatorname{erfc}\left(\sqrt{\alpha s} + \frac{\beta}{2\sqrt{\alpha}}\right)\right\} =$$

$$= \frac{\sqrt{\alpha}}{\pi\sqrt{\tau}(\tau+\alpha)} \exp\left[-\left(\frac{1}{\alpha} + \frac{1}{\tau}\right)\frac{\beta^2}{4}\right] U(\sqrt{\tau} - \beta),$$

where $U(X)$ is the symmetric unit function [5].

For $z = 0$

$$L^{-1}\left\{\exp\left(\frac{r_0^2 s}{4a}\right)\operatorname{erfc}\left(\frac{r_0\sqrt{s}}{2\sqrt{a}}\right)\right\} = \frac{r_0}{2\pi\sqrt{a\tau}} \frac{1}{\left(\tau + \frac{r_0^2}{4a}\right)}.$$

For the central point ($r = z = 0$) of the heated spot we have

$$\Theta_1(0,\ 0,\ \tau) = \frac{\rho r_0^2}{4\sqrt{\pi}\lambda\sqrt{a}} \int_0^\tau \frac{w_0(\tau-\xi)}{\sqrt{\xi}\left(\xi + \frac{r_0^2}{4a}\right)} \times \left[1 - \frac{1}{e}\exp\left(-\frac{r_0^2}{4a\xi}\right)\right] d\xi,$$

where

$$1/e = \exp(-1). \tag{18}$$

The expression to determine the temperature $\Theta_1(r,\ 0,\ \tau) = T_1(r,\ 0,\ \tau) - T_0$ inside the heated spot ($0 \leq r < r_0$) of the body surface ($z = 0$) has the form:

$$\Theta_1(r,\ 0,\ \tau) = \frac{\rho}{b\sqrt{\pi}} \exp\left(-\frac{r^2}{r_0^2}\right) \int_0^\tau \frac{w_0(\tau-\xi)}{\sqrt{\xi}} d\xi -$$

$$- \sqrt{\frac{2}{\pi}} \frac{\rho}{eb} \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{n,m} \left(\frac{r}{r_0}\right)^{2n} \left(\frac{r_0}{\sqrt{a}}\right)^{2n-m-\frac{1}{2}} \times$$

$$\times \int_0^\tau w_0(\tau-\xi)\, \xi^{-n+\frac{m}{2}-\frac{1}{4}} \exp\left(-\frac{r_0^2}{8a\xi}\right) W_{n-\frac{m}{2}+\frac{1}{4},\, -\frac{m}{2}+\frac{1}{4}}\left(\frac{r_0^2}{4a\xi}\right) d\xi -$$

$$- \frac{1}{\sqrt{\pi}} \frac{\rho}{b r_0^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} B_{n,m} \left(\frac{r^2}{a}\right)^n \int_0^\tau w_0(\tau-\xi)\, \xi^{-n-\frac{1}{2}} \times \tag{19}$$

$$\times \int_{r^2}^{r_0^2} \exp\left[-\left(\frac{1}{r_0^2} + \frac{1}{4a\xi}\right) x\right] L_{n-m}^{\left(m-\frac{1}{2}\right)}\left(\frac{x}{4a\xi}\right) dx d\xi +$$

$$+ \sqrt{\frac{2}{\pi}} \frac{\rho}{b r_0^2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{A_{n,m}}{(n+1)} r^{-m-\frac{3}{2}} (\sqrt{a})^{-2n+m-\frac{1}{2}} \times$$

$$\times \int_0^\tau w_0(\tau-\xi)\, \xi^{-n+\frac{m}{2}-\frac{3}{4}} \exp\left(-\frac{r^2}{8a\xi}\right) W_{n-\frac{m}{2}+\frac{3}{4},\, -\frac{m}{2}-\frac{1}{4}}\left(\frac{r^2}{8a\xi}\right) \times \int_0^r x^{2n+3} \exp\left(-\frac{x^2}{r_0^2}\right) dx d\xi,$$

where

$$B_{n,m} = (-1)^{m-n} \frac{(2m-1)!!}{2^m 4^n n! m!};\quad A_{n,m} = \frac{(2m-1)!!}{4^n n! m! (n-m)!};$$

$$\int_0^r x^{2n+3}\exp\left(-\frac{x^2}{r_0^2}\right)dx=\frac{r_0^{2n+4}}{2}\gamma\left(n+2,\ \frac{r^2}{r_0^2}\right);\ b=\frac{\lambda}{\sqrt{a}};$$

$$\int_{r^2}^{r_0^2}\exp\left[-\left(\frac{1}{r_0^2}+\frac{1}{4a\xi}\right)x\right]L_{n-m}^{\left(m-\frac{1}{2}\right)}\left(\frac{x}{4a\xi}\right)dx=$$

$$=\frac{1}{\left(\frac{1}{r_0^2}+\frac{1}{4a\xi}\right)}\sum_{k=0}^{n-m}\frac{(-1)^k}{k!}\binom{n-\frac{1}{2}}{n-m-k}\left(\frac{1}{\frac{4a\xi}{r_0^2}+1}\right)^k\times\left[\gamma\left(k+1,\ 1+\frac{r_0^2}{4a\xi}\right)-\gamma\left(k+1,\ \frac{r^2}{r_0^2}+\frac{r^2}{4a\xi}\right)\right]; \tag{20}$$

and $W_{k,\mu}(X)$, $L_k^{\alpha}(X)$, $\gamma(\alpha, X)$ are, respectively, the Whittaker, multiterm Laguerre, and incomplete gamma functions [4, 6-8].

The expression to determine the temperature $\Theta_2(r, 0, \tau) = T_2(r, 0, \tau) - T_0$ outside the heated spot ($r > r_0$) on the body surface ($z = 0$) can be written in the form:

$$\Theta_2(r,\ 0,\ \tau)=\frac{1}{\sqrt{2\pi}}\frac{\rho}{eb}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{A_{n,m}}{n+1}\left(\frac{r_0}{r}\right)^{m+\frac{3}{2}}\times$$
$$\times\left(\frac{r_0}{\sqrt{a}}\right)^{2n-m+\frac{1}{2}}\int_0^{\tau}w_0(\tau-\xi)\,\xi^{-n+\frac{m}{2}-\frac{3}{4}}\exp\left(-\frac{r^2}{8a\xi}\right)\times$$
$$\times W_{n-\frac{m}{2}+\frac{3}{4},-\frac{m}{2}-\frac{1}{4}}\left(\frac{r^2}{4a\xi}\right)d\xi+\frac{1}{\sqrt{2\pi}}\frac{\rho}{b}\times \tag{21}$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{A_{n,m}}{n+1}\left(\frac{r_0}{r}\right)^{m+\frac{3}{2}}\left(\frac{r_0}{\sqrt{a}}\right)^{2n-m+\frac{1}{2}}\times$$
$$\times\gamma\left(n+2,\ \frac{r^2}{r_0^2}\right)\int_0^{\tau}w_0(\tau-\xi)\,\xi^{-n+\frac{m}{2}-\frac{3}{4}}\exp\left(-\frac{r^2}{8a\xi}\right)\times W_{n-\frac{m}{2}+\frac{3}{4},-\frac{m}{2}-\frac{1}{4}}\left(\frac{r^2}{4a\xi}\right)d\xi.$$

The solutions (17)-(21) are written in the form of quadratures on an infinite interval. By assigning the form of the function $w_0(\tau)$ and carrying out the uncomplicated integration of the expressions obtained we shall obtain a series of partial solutions for the temperature fields $\Theta_i(r, z, \tau)$ ($i = 1, 2$) in the semiinfinite body (in the thermal sense) heated by a laser source. We postulate that $w_0(\tau) = w_0 = \mathrm{const}$. Then the expressions obtained above for the temperature fields $\Theta_i(r, z, \tau)$ have the following form:

for $r = 0$, $z + 0$, $\tau > 0$

$$\frac{\Theta_1^*(0,\ z,\ \mathrm{Fo})}{\mathrm{Ki}}=\frac{\rho}{\sqrt{\pi}}\left\{\mathrm{arctg}(2\sqrt{\mathrm{Fo}})\exp\left(-\frac{z^2/r_0^2}{4\mathrm{Fo}}\right)\times\right.$$
$$\left.\times\left[1-\frac{1}{e}\exp\left(-\frac{1}{4\mathrm{Fo}}\right)\right]-\int_0^{4\mathrm{Fo}}\xi^{-2}\,\mathrm{arctg}(\sqrt{\xi})\exp\left(-\frac{z^2/r_0^2}{\xi}\right)\times\left[\frac{z^2}{r_0^2}-\frac{1}{e}\left(1+\frac{z^2}{r_0^2}\right)\exp\left(-\frac{1}{\xi}\right)\right]d\xi\right\}; \tag{22}$$

for $r = z = 0$ (Fo > 0)

$$\frac{\Theta_1^*(0,\ 0,\ \mathrm{Fo})}{\mathrm{Ki}}=\frac{\rho}{\sqrt{\pi}}\left\{\mathrm{arctg}(2\sqrt{\mathrm{Fo}})\left[1-\frac{1}{e}\exp\left(-\frac{1}{4\mathrm{Fo}}\right)\right]+\frac{1}{e}\int_0^{4\mathrm{Fo}}\xi^{-2}\,\mathrm{arctg}(\sqrt{\xi})\exp\left(-\frac{1}{\xi}\right)d\xi\right\}, \tag{23}$$

where

$$\mathrm{Ki} = \frac{w_0 r_0}{\lambda T_0}; \quad \mathrm{Fo} = \frac{a\tau}{r_0^2}; \quad \Theta_1^*(0,\ 0,\ \mathrm{Fo}) = \frac{\Theta_1(0,\ 0,\ \tau)}{T_0}.$$

A graph of Eq. (23) is shown in Fig. 1.

Calculation of an integral of the form $\int\limits_0^{4\mathrm{Fo}} \xi^{-2} \operatorname{arctg}(\sqrt{\xi}) \exp\left(-\frac{1}{\xi}\right) d\xi$ is not difficult to carry out, expanding ($\sqrt{\xi}$) in the series:

$$\int\limits_0^{4\mathrm{Fo}} \xi^{-2} \operatorname{arctg}(\sqrt{\xi}) \exp\left(-\frac{1}{\xi}\right) d\xi =$$

$$= \begin{cases} \sum\limits_{n=0}^{\infty} (-1)^n \frac{(4\mathrm{Fo})^{\frac{n}{2}+\frac{1}{4}}}{2n+1} \exp\left(-\frac{1}{8\mathrm{Fo}}\right) W_{-\frac{n}{2}-\frac{1}{4},\ \frac{n}{2}-\frac{1}{4}}\left(\frac{1}{4\mathrm{Fo}}\right) \\ \text{for} \quad 2\sqrt{\mathrm{Fo}} \leqslant 1; \\ \frac{\pi}{2} \exp\left(-\frac{1}{4\mathrm{Fo}}\right) + \sum\limits_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1} (4\mathrm{Fo})^{-\frac{n}{2}-\frac{1}{4}} \times \\ \times \exp\left(-\frac{1}{8\mathrm{Fo}}\right) W_{\frac{n}{2}+\frac{1}{4},\ -\frac{n}{2}-\frac{3}{4}}\left(\frac{1}{4\mathrm{Fo}}\right) \quad \text{for} \quad 2\sqrt{\mathrm{Fo}} > 1. \end{cases} \tag{24}$$

The distribution of relative temperature outside the heated spot ($r > r_0$, $z = 0$) on the body surface can be written in the form:

$$\frac{\Theta_2^*(r,\ 0,\ \tau)}{\mathrm{Ki}} = \frac{\rho}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{A_{n,m}}{n+1} \left(\frac{r_0}{r}\right)^{m+\frac{3}{2}} \left(\frac{1}{\sqrt{\mathrm{Fo}}}\right)^{2n-m-\frac{1}{2}} \times$$
$$\times \left[\frac{1}{e} + \gamma\left(n+2,\ \frac{r^2}{r_0^2}\right)\right] \exp\left(-\frac{r^2}{r_0^2}\frac{1}{8\mathrm{Fo}}\right) \times$$
$$\times W_{n-\frac{m}{2}-\frac{1}{4},\ \frac{m}{2}+\frac{1}{4}}\left(\frac{r^2}{r_0^2}\frac{1}{4\mathrm{Fo}}\right),$$

where

$$\gamma\left(n+2,\ \frac{r^2}{r_0^2}\right) = (n+1)!\left[1 - \exp\left(-\frac{r^2}{r_0^2}\right) \sum_{k=0}^{n+1} \frac{1}{k!} \left(\frac{r^2}{r_0^2}\right)^k\right]. \tag{25}$$

On the basis of the dependences obtained for the temperature fields in a semiinfinite body heated by a laser source we can propose a number of methods of determining the thermophysical characteristics and also the absorptance of the body (for known values of the thermophysical properties), if in a thermophysical experiment we accomplish local heating of the body surface by a laser source (up to temperature values in the vicinity of the heated spot which do not cause disintegration (removal of mass) and do not cause nonlinearity of the thermophysical properties). In regard to possible values of the laser radiative power, it may vary over a wide range from several milliwatts to tens of kilowatts. Some types of CO_2 lasers can achieve power of tens of kilowatts, operating in the continuous regime, and the peak power of pulsed solid-state lasers reaches 10^{12} Watt [2].

From the viewpoint of using lasers in a thermophysical experiment to create a directional heat flux there is no real possibility of reaching high power levels, but it is possible to control this power, i.e., the beam power, referred to unit area, of radiation falling on the surface. By focusing coherent laser radiation by means of lenses and mirrors we can achieve very diverse values of the radiative power density directed to the surface of a nontransparent test surface.

The simplest method of determining the thermophysical properties and also the absorptance of a body (or the reflectance $(1 - \rho)$ is based on a knowledge from experiment of the excess temperature $\Theta_1(0,\ 0,\ \tau)$ of a function of time τ at the central point ($r = z = 0$) of the heated spot in the initial time of action (switching on) of the laser.

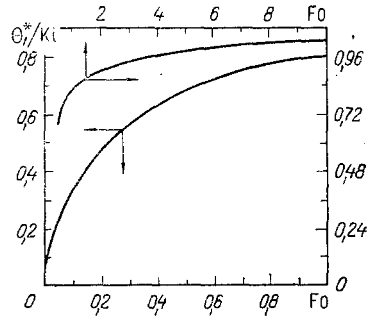

Fig. 1. Graph of the dependence $\Theta_1^*(0, 0, \mathrm{Fo})/\mathrm{Ki} = f(\mathrm{Fo})$, Eq. (23), for $\rho = 1$ for the central point of the heated spot ($r = z = 0$).

For Fo $\leq$ 0.08 (or more exactly, for Fo $\to$ 0) the value of the relative excess temperature at the center of the heated spot can be written in the form

$$\frac{\Theta_1^*(0,\ 0,\ \mathrm{Fo})}{\mathrm{Ki}} = \frac{\rho}{\sqrt{\pi}}\,\mathrm{arctg}\,(2\sqrt{\mathrm{Fo}}). \tag{26}$$

The last expression can be written more simply if we bear in mind that for small Fo the value of the function $(2\sqrt{\mathrm{Fo}}) \approx 2\sqrt{\mathrm{Fo}}$.

Thus, for small Fo the dependence of excess temperature at the center of the heated spot of the surface of a nontransparent semiinfinite surface, heated by a laser source (of constant power) corresponds to the classical formula for determining the desired temperature with heating of a massive body by an arc heat source of constant power (with $\rho = 1$) [4, 9, 10]:

$$T_1(0,\ 0,\ \tau) - T_0 = \Theta_1(0,\ 0,\ \tau) = \frac{2\rho w_0\sqrt{\tau}}{b\sqrt{\pi}}, \tag{27}$$

where $b = \lambda/\sqrt{a}$ is the thermal activity of the nontransparent body.

Depending on the knowledge of the physical (or thermophysical) properties of the test object (presence of specific data on thermophysical properties, reflectance or absorptance) on the basis of the formulas presented there are numerous possible variants (i.e., methods) for calculating the desired parameters. By way of example we shall consider some of them. If we know the thermal activity of the body and the value of the radiative intensity w_0 (W/m²), then the absorptance is $\rho = b\sqrt{\pi}\Theta_1(0, 0, \tau)/(2w_0\sqrt{\tau})$. For known values of ρ and w_0 the value of the thermal activity can be calculated from the formula $b = 2\rho\pi^{-1/2}w_0\sqrt{\tau}/\Theta_1(0, 0, \tau)$.

If we start from Eq. (26), then for known values of the Ki and Fo numbers the absorptance is $\rho = \Theta_1^*(0, 0, \tau)\sqrt{\pi}\mathrm{Ki}^{-1}/\mathrm{arctg}\,(2\sqrt{\mathrm{Fo}})$. For known values of Fo and ρ we can determine the value of the Kirpichev number $\mathrm{Ki} = \sqrt{\rho\pi^{-1}}\Theta_1^*(0, 0, \mathrm{Fo})/\mathrm{arctg}\,(2\sqrt{\mathrm{Fo}})$. For an obtained value of Ki and known w_0, r_0, T_0, we can determine the thermal conductivity from the formula $\lambda = w_0 r_0/(\mathrm{Ki}T_0)$. There are possibly other variants for determining the thermophysical properties and ρ, both from Eqs. (23), (26) and (27), and by using the laws for development of the excess temperatures for $\Theta_1(0, z, \tau)$ and $\Theta_2(r, 0, \tau)$, respectively, on the axis ($r = 0$) of a semiinfinite body and outside the heated spot ($r > r_0$) on the surface of a nontransparent body ($z = 0$).

NOTATION

$W_0(\tau)$, laser source power (W); $w_0(r, \tau)$, $w_0(\tau)$, intensity (density) of the laser radiation (W/m²); ρ, absorptance of the nontransparent body; $\overline{\Theta}_{ic}(r, p, s)$ representation of the excess temperatures in the corresponding regions of variation of the cylindrical coordinate ($i = 1, 2$); p, s, respectively, the parameters of the infinite integral Fourier cosine and Laplace transformations; r_0, radius of a Gaussian laser beam on the surface of the nontransparent body; $I_0(x)$, $K_0(x)$, $I_1(x)$, $K_1(x)$, modified Bessel functions of the appropriate order; r, z, and τ, cylindrical coordinates and time; $\overline{w}_0(s)$, Laplace-transformed value of the specific intensity (per unit area) of the laser source; $e = \exp(+1)$; a, b, λ, thermal diffusivity, thermal activity, and thermal conductivity of the nontransparent body; $\Theta_1(r, z, \tau) = T_1(r, z, \tau)$ —

T_0, $\Theta_2(r, z, \tau) = T_2(r, z, \tau) - T_0$, excess temperatures; T_0, initial temperature; erf (x), probability integral; $C_{2n}^m = \frac{(2n)!}{m!\,(2n-m)!}$, binomial coefficients; $(3/2)_n = 2^{-2n}\cdot\frac{(2n+1)!}{n!}$, Pokhgammer symbol; $D_\nu(x)$, parabolic cylinder function; $A_{n,m}$, $B_{n,m}$, thermal amplitudes (from the text); $H_k(x)$, orthogonal Hermite polynomials; w_0, content (in time) density of radiation (W/m^2); $\gamma(n + 2, r^2/r_0^2)$, incomplete gamma function; $W_{k,\mu}(x)$, Whittaker function; $Ki = w_0 r_0/(\lambda T_0)$, $Fo = a\tau/r_0^2$, Kirpichev and Fourier numbers; $\Theta_1^*(0, 0, \tau)$, $\Theta_2^*(r, 0, \tau)$, dimensionless relative temperatures; $L_k^\alpha(x)$, Laguerre polynomial.

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NUMERICAL ANALYSIS OF FUNCTIONALLY INTEGRATED VLSIC ELEMENTS TAKING INTO ACCOUNT HEAT EFFECTS. II. METHOD AND PROGRAM

I. I. Abramov and V. V. Kharitonov

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The program and method of implementation of a discrete, multidimensional physical-topological model, taking into account heat effects, are described.

After analyzing construction of a discrete physical-topological model of functionally integrated VLSIC elements taking into account heat effects [1] we shall now present a method for implementing it and we shall describe a universal program.

Method for Selecting the Starting Approximation. The method is based on the solution of a truncated system of equations derived from the starting system (Eqs. (1)-(8) from [1]) with the help of a number of physical assumptions. The key assumption is the assumption that the temperature is constant over the structure of an element. This means that self-heating of the element is neglected in the starting approximation. As a result, Eq. (6) or [1] need not be solved.

The effect of the temperature of the surrounding medium must, however, be taken into account. Because of the adopted physical assumptions the equations for the current densities can be written in a different form:

$$\mathbf{j}_p = -q\mu_p(T_{0c})\,p\nabla\Phi_p, \tag{1}$$

$$\mathbf{j}_n = -q\mu_n(T_{0c})\,n\nabla\Phi_n. \tag{2}$$

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